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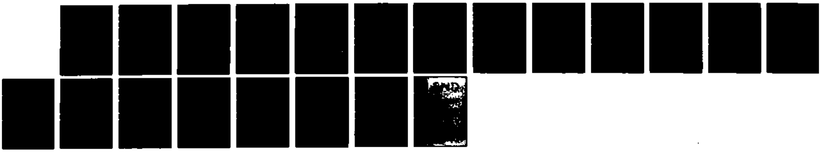

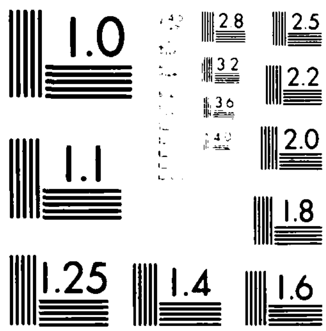

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CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
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ON THE PROBABILITY GENERATING FUNCTIONAL FOR POINT PROCESSES

by

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and

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An extended probability generating functional (p.g.fl.) $\bar{G}[h] = E(\exp \int_X \log h(x) N(dx))$ is well-defined for any point process N on the complete separable metric space X over the space $\bar{V}_0$ of measurable functions h: X → [0,1] such that $\inf_{x \in X} h(x) > 0$. The distribution of N is determined uniquely by p.g.fl. G[h] over the smaller space V_0 of functions $h \in \bar{V}_0$ for which 1-h has bounded support. Continuity [illegible] $\bar{V}$ [illegible] pointwise convergence [illegible] h_n [illegible] V or $\bar{V}_0$ or V [illegible] [0,1] [illegible]

V = {h ∈ V̄: 1-h has bounded support} [illegible]

p.g.fl. proof of the mixing property of certain stationary cluster processes.

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ON THE PROBABILITY GENERATING FUNCTIONAL FOR POINT PROCESSES

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Summary

An extended probability generating functional (p.g.fl.) $\bar{G}[h] = E(\exp \int_{\mathcal{X}} \log h(x)\, N(dx))$ is well-defined for any point process N on the complete separable metric space $\mathcal{X}$ over the space $\bar{\mathcal{V}}_0$ of measurable functions $h : \mathcal{X} \to (0,1]$ such that $\inf_{x \in \mathcal{X}} h(x) > 0$. The distribution of N is determined uniquely by the p.g.fl. $G[h] \equiv \bar{G}[h]$ over the smaller space $\mathcal{V}_0$ of functions $h \in \bar{\mathcal{V}}_0$ for which $1-h$ has bounded support. Continuity results for $\bar{G}[.]$ involving pointwise convergent sequences $\{h_n\} \subset \mathcal{V}_0$ or $\bar{\mathcal{V}}_0$ or $\bar{\mathcal{V}} = \{$measurable $h : \mathcal{X} \to [0,1]\}$ or $\mathcal{V} = \{h \in \bar{\mathcal{V}} : 1-h$ has bounded support$\}$ are reviewed, and used in furnishing a complete p.g.fl. proof of the mixing property of certain stationary cluster processes.

Keywords: Continuity of generating function, mixing of point process, extended probability generating functional.

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1. Introduction

Various authors, in particular Moyal (1962) and Westcott (1972), have developed the discussion of point processes via probability generating functionals (p.g.fl.s). The object of this paper is to collect together some notes concerning the spaces of functions on which p.g.fl.s may be defined, introduce the extended p.g.fl., and use this extended p.g.fl. to establish mixing properties of stationary cluster processes via p.g.fl. techniques.

We work with point processes defined on some complete separable metric space X . $\hat{N}_X$ denotes the space of counting measures defined on the Borel subsets $\mathcal{B}_X$ of X such that these measures are finite on bounded sets in $\mathcal{B}_X$. This set-up corresponds with that of Mathes, Kerstan and Mecke (1978); Kallenberg (1975) assumes that X is locally compact as well. In measure-theoretic language, a point process N is a measurable mapping of a probability space $(\Omega, \mathcal{F}, P)$ into $(\hat{N}_X, \mathcal{B}(\hat{N}_X))$ where $\mathcal{B}(\hat{N}_X)$ is the smallest σ-algebra with respect to which the mappings $N \to N(A)$ are measurable for each $A \in \mathcal{B}_X$.

The appropriate transform tool for the discussion of a random measure ξ defined on X , as distinct from a random signed measure, is the <u>Laplace functional</u> $L[f]$ defined on the space $BM_+(X)$ of bounded measurable non-negative functions f of bounded support (i.e., vanishes outside some bounded set in $\mathcal{B}_X$) by

$$(1.1) \qquad L[f] = E\exp\left(-\int_X f(x)\xi(dx)\right) .$$

A1

Kallenberg (1975, p.6) calls this functional the L-transform, denoting it by L_{ξ} . He shows (his Theorem 3.1) that the distribution of a random measure on a locally compact space is determined by the values of $L[f]$ for f in the smaller space $F_c \subset BM_+(X)$ of non-negative continuous functions on X with compact support. As a corollary, we deduce that a point process, being a special case of a random measure, has its distribution determined by the values of $L[f]$ for $f \in BM_+(X)$. (Strictly speaking, this is not a corollary unless X is locally compact; rather we have a corollary to the extension of Kallenberg's theorem to the case of X a complete separable metric space.)

However, the most convenient transform tool for the discussion of a point process N is not the Laplace functional, but, by analogy with the probability generating function (p.g.f.) for non-negative integer-valued random variables (r.v.s), the probability generating functional (p.g.fl.) $G[.]$. Here it is not quite so clear as to which space is the most appropriate one to use in the definition, and the main theme of this note is that the choice of a slightly smaller space than has been customary simplifies and clarifies the proofs of some useful theorems on mixing properties and related topics.

We define the p.g.fl. first on the space U of measurable complex-valued functions g on X of modulus ≤ 1 and for which $1-g$ has bounded support : set

$$(1.2) \qquad G[g] = E \exp \int_X \log g(x) N(dx)$$

with the convention that $\exp \int_X \log g(x)N(dx) = 0$ if

$N(\{x\}) > 0$ for any x for which $g(x) = 0$. In this (first) definition of $G[.]$, the use of exponentiation, coupled with $\exp(2\pi i) = 1$ and the fact that $N(\cdot)$ is atomic with integer-valued atoms, overcomes any possible ambiguity that could come from the use of different branches of $\log g(.)$ for complex, non-zero, $g(.)$, while the possible indeterminacy that could come from $\log g(x)$ when $g(x) = 0$ is obviated by the caveat. Since the very reason for the use of an integral in (1.2) is to by-pass the question of defining what may possibly be an infinite product, the caveat resolves the possible indeterminacy in a manner consistent with the purpose of the definition.

The space $\mathcal{U}$ is certainly richer than needed. Ideally, the choice of a function space used in defining transforms is dictated by reasons of convenience and economy, the former requiring the space to be large enough to admit closure under (e.g.) limit operations, the latter requiring the space to be small enough (consistent with uniqueness of determination of the distribution of the random entity involved) so as to allow the maximum flexibility in manipulation. Westcott (1972) principally used for $G[.]$ not $\mathcal{U}$ but the space $\mathcal{V} \equiv \mathcal{V}(X)$ of $[0,1]$-valued measurable functions h for which $1-h$ has bounded support. (In fact, Westcott used V to denote the space of $[0,1]$-valued measurable functions with bounded support, so any measurable $g : X \to [0,1]$ has $g \in \mathcal{V}$ iff $1-g \in V$. Thus, the distinction between $\mathcal{V}$ and V is essentially a notational convention, but we believe the use of $\mathcal{V}$ to be preferable because it enables theorems to be stated more economically.)

The use of the space

(1.3) $$\mathcal{V}_0 \equiv \mathcal{V}_0(\mathcal{X}) \equiv \{h \in \mathcal{V} : \inf_{x \in \mathcal{X}} h(x) > 0\}$$

has certain advantages, especially in the statement of continuity results. Moreover, $\mathcal{V}_0$ is the natural counterpart of $BM_+(\mathcal{X})$ since

(1.4) $$-\log h(.) \in BM_+(\mathcal{X}) \quad \text{iff} \quad h(.) \in \mathcal{V}_0 .$$

It thus follows from the statement concerning Laplace functionals that the distribution of a point process is determined by $\{G[h] : h \in \mathcal{V}_0\}$.

2. The extended p.g.fl. and continuity properties

Neither $\mathcal{V}_0$ nor $\mathcal{V}$ is closed under pointwise convergence. Such closure is a property of the richer class of functions

$$\bar{\mathcal{V}} \equiv \{\text{measurable } h : \mathcal{X} \to [0,1]\}.$$

Moreover, given any $h \in \bar{\mathcal{V}}$, there exists a sequence of functions $\{h_n\} \subset \mathcal{V}_0$ such that

(2.1) $$h_n(x) \to h(x) \qquad (n \to \infty) ,$$

because, for any monotone increasing sequence of bounded sets A_n with limit $\mathcal{X}$, the relation

$$h_n(x) = 1 - I_{A_n}(x)(1-n^{-1})(1-h(x)) \qquad (n = 1,2,\ldots)$$

yields such a sequence.

We shall also have use for

$$\bar{\mathcal{V}}_0 \equiv \{h \in \bar{\mathcal{V}} : \inf_{x \in \mathcal{X}} h(x) > 0\} .$$

Now, given any $h \in \bar{V}$, the integral

$$\int_X \log h(x)\ N(dx)$$

is uniquely defined as a countable sum of non-positive number, whether the resulting quantity is finite or infinite. Let $\{h_n\} \subset V_0$ satisfy (2.1). Then the integrals

$$\int_X \log h_n(x)\ N(dx) \tag{2.2}$$

are finite a.s., and the monotonicity of h_n enables us to apply the Lebesgue monotone convergence theorem to conclude that as $n \to \infty$

$$\int_X \log h_n(x)\ N(dx) \to \int_X \log h(x)\ N(dx) \quad \text{a.s.,} \tag{2.3}$$

whether the limit is finite or infinite. Since the exponential of each term in (2.3) is bounded by unity, an appeal to the dominated convergence theorem shows that

$$G[h_n] \equiv E(\exp \int_X \log h_n(x) N(dx)) \tag{2.4}$$

$$\to E(\exp \int_X \log h(xO\ N(dx)) \equiv \bar{G}[h]$$

where the right-hand side is taken as the definition of the <u>extended p.g.fl.</u> $\bar{G}[h]$ over $h \in \bar{V}$.
(It is a simple exercise to verify that, if $\{h_n'\} \subset V_0$ is any other sequence that converges monotonically to h pointwise as at (2.1), then $G[h_n'] \to \bar{G}[h]$ as at (2.4), i.e., $\bar{G}[h]$ is well-defined via the construction involving sequences $\downarrow h$.)

The same argument can be applied to $\bar{G}[.]$ itself whenever $\{h_n\} \subset \bar{V}$ satisfies (2.1), and so leads to part (ii) of the following compendium of continuity results concerning $\bar{G}[.]$.

THEOREM 2.1 <u>Let</u> N <u>be any given point process and</u> $\{h_n\}$ <u>a pointwise convergent sequence of functions</u> $h_n \in \mathcal{V}$ <u>with limit</u> h . <u>The extended p.g.fl.</u> $\bar{G}$ <u>of</u> N <u>satisfies</u>

$$(2.5) \qquad \bar{G}[h_n] \to \bar{G}[h] \qquad \text{as} \quad n \to \infty$$

<u>whenever one of the following holds</u>:

(i) $N(\mathcal{X}) < \infty$ a.s.

(ii) $h_n(x) \downarrow h(x)$ (all x) .

(iii) $h_n(x) \uparrow h(x)$ (all x) <u>and</u> $\{h_n\} \subset V$.

(iv) $|\log(h_n(x)/h(x))| \le \varepsilon(x)$ (all n) <u>where the function</u> $\varepsilon(.)$ <u>is measurable and satisfies</u>

$$(2.6) \qquad \int_{\mathcal{X}} \varepsilon(x) N(dx) < \infty \qquad \text{a.s.}$$

(v) <u>For all sufficiently large</u> n , $h_n(x) \ge h_0(x)$ (all x) <u>for some</u> $h_0 \in \mathcal{V}_0$, <u>and</u> $|h_n(x) - h(x)| \le \varepsilon(x)$ (all x) <u>for some measurable function</u> $\varepsilon(.)$ <u>satisfying</u> (2.6).

<u>Remark</u>. This compendium could be extended, as in Westcott's (1972) Theorem 2, by including conditions involving the first moment measure $M(.) = E\,N(.)$, assuming the existence of $M(.)$ (i.e., finiteness of $M(A)$ for bounded $A \in \mathcal{B}_{\mathcal{X}}$) .

<u>Proof</u>. When condition (i) holds, there exists, except possibly

on a $\mathcal{P}$-null set, a finite set of points $x_1(\omega),\dots,x_{N(X)}(\omega)$ such that

$$\exp\int_X \log h_n(x)N(dx) = \prod_{i=1}^{N(\omega)} h_n(x_i(\omega)) \to \prod_{i=1}^{N(\omega)} h(x_i(\omega))$$

by the pointwise convergence of h_n, i.e., (2.3) holds. Then by dominated convergence, much as at (2.4), we must have (2.5).

We have already indicated the proof that (ii) implies (2.5). When (iii) holds, it follows that the integrals at (2.3) are finite and the convergence is monotone, so, again, (2.4) holds. (Indeed, under conditions (iii), $h \in V$ also; if also $h_n \in V_0$, then $h \in V_0$ also.)

Introduce $h_n'(x) = \min(h_n(x),h(x))$, $h_n''(x) = \max(h_n(x),h(y))$, so that when (iv) holds, $0 \le \log(h_n''(x)/h_n'(x)) \le \varepsilon(x)$, and therefore, by dominated convergence,

$$(2.7) \qquad 0 \le \int_X \log(h_n''(x)/h_n'(x))\, N(dx) \to 0 \quad \text{a.s.} \qquad (n \to \infty) .$$

Consequently we then have

$$\begin{aligned}|\bar{G}[h_n] - \bar{G}[h]| &= |E(\exp\int_X \log h_n(x)N(dx) - \exp\int_X \log h(x)N(dx))| \\ &\le |E\{(\exp\int_X \log h_n''(x)N(dx)) \\ &\qquad\qquad (1 - \exp\int_X \log [h_n'(x)/h_n''(x)]N(dx))\}| \\ &\le 1 - E\{\exp\int_X [-\log(h_n''(x)/h_n'(x))]N(dx)\} \to 0 \qquad (n \to \infty)\end{aligned}$$

by dominated convergence with respect to expectation (i.e., the $\mathcal{P}$-integration) using the pointwise convergence result at (2.7).

When (v) holds, suppose without essential loss of generality that for all n, $h_n(x) \geq h_0(x)$ (all x). Then $\inf_{x \in X} h_n(x) \geq \inf_{x \in X} h_0(x) \equiv \gamma$ say, with $\gamma > 0$, and thus, with h_n', h_n'' as in the proof with (iv) holding,

$$|\log(h_n(x)/h(x))| = |\log[1 - (1 - h_n'(x)/h_n''(x))]|$$

$$\leq (1 - h_n'(x)/h_n''(x)) \sum_{j=0}^{\infty} (1-\gamma)^j$$

$$= \gamma^{-1}(1 - h_n'(x)/h_n''(x))$$

$$\leq \gamma^{-2}|h(x) - h_n(x)| \leq \gamma^{-2}\varepsilon(x).$$

We can now mimic the proof as for part (iv).

The continuity at $s = 1-0$ of a p.g.f. $\phi(s) = E\, s^N$ for an a.s. finite non-negative integer-valued r.v. N, implies the continuity for all $0 < s < 1$ because, with $0 \leq s_1 \leq 1$, $\phi(s_1 s)/\phi(s)$ is again a p.g.f. The analogous statement for point processes requires some care in formulating the necessary qualifications.

THEOREM 2.2. Let $\{h_n\} \subset \mathcal{V}$ be a pointwise convergent sequence with $\lim_{n\to\infty} h_n(x) = 1$ for every $x \in X$. Then for a point process N with extended p.g.fl. $\bar{G}[.]$,

$$\bar{G}[h_n] \to 1 \qquad \text{as } n \to \infty \tag{2.8}$$

if and only if for every $h \in \mathcal{V}$,

$$\bar{G}[h_n h] \to \bar{G}[h] \qquad \text{as } n \to \infty . \tag{2.9}$$

Proof. In one direction the theorem is trivial, because (2.9) reduces to (2.8) with $h(x) \equiv 1$. For the converse, we have merely to observe that

$$\bar{G}[h] = EX, \quad \bar{G}[h_n h] = EXY_n, \quad \bar{G}[h_n] = EY_n$$

for certain $[0,1]$-valued r.v.s X, Y_n, and thus

$$0 \le \bar{G}[h] - \bar{G}[hh_n] = E(X(1-Y_n)) \le E(1-Y_n) = 1-\bar{G}[h_n] ,$$

showing that (2.8) implies (2.9).

Another p.g.fl. analogue of the continuity property of p.g.f.s is the following result.

THEOREM 2.3 <u>For</u> $h \in \mathcal{V}$, $\bar{G}[1-\alpha(1-h)] \to 1$ <u>as</u> $\alpha \downarrow 0$ <u>if and only if</u> $\int_{\mathcal{X}} (1-h(x))N(dx) < \infty$ a.s.

<u>Proof</u>. For fixed α in $(0,1)$, since $0 \le 1-h(x) \le 1$ $(x \in \mathcal{X})$,

$$-\alpha(1-h) \ge \log(1-\alpha(1-h)) \ge \alpha(1-h) \sum_{k=0}^{\infty} (\alpha(1-h))^k$$

$$\ge -\alpha(1-\alpha)^{-1}(1-h) .$$

Thus, $$E\exp(-\alpha\int_{\mathcal{X}}(1-h(x))N(dx)) \ge \bar{G}[1-\alpha(1-h)]$$

$$\ge E\exp(-\alpha(1-\alpha)^{-1}\int_{\mathcal{X}}(1-h(x))N(dx)) .$$

By elementary properties of Laplace-Stieltjes transforms of $[0,\infty]$-valued r.v.s, it follows that the first and last terms in these inequality relations converge as $\alpha \downarrow 0$ to $\Pr\{\int_{\mathcal{X}}(1-h(x))N(dx) < \infty\}$.

3. Mixing of point processes

Stationarity of a point process N on $\mathcal{X} = \mathbb{R}^d$ is the requirement that the joint distributions of $T_x N(.) \equiv N(.+x)$ be independent of x, or, by extension, that $\mathcal{P}(T_x U) = \mathcal{P}(U)$ for all $x \in \mathbb{R}^d$ and $U \in \mathcal{F}$. A stationary point process $N(.)$ is said to be mixing when for any $U, V \in \mathcal{F}$,

$$\mathcal{P}(U \cap T_x V) \to \mathcal{P}(U)\mathcal{P}(V) \quad \text{as} \quad \|x\| \to \infty .$$

Westcott (1972) showed in the case $d = 1$, and his proof carries over unchanged to the case of general finite integer d, that $N(.)$ is mixing iff its p.g.fl. $G[.]$ satisfies, for all $h_1, h_2 \in \mathcal{V}$,

$$(3.1) \qquad G[h_1 \; T_x h_2] \to G[h_1]G[h_2] \quad \text{as} \quad \|x\| \to \infty$$

where for any function $h : \mathbb{R}^d \to \mathbb{R}$, $T_x h(u) = h(u+x)$. Recalling that $\mathcal{P}(.)$ is determined uniquely by $\{G[h] : h \in \mathcal{V}_0\}$, we have the following seemingly trivial modification of Westcott's result.

THEOREM 3.1. <u>A stationary point process $N(.)$ on $\mathbb{R}^d$ is mixing iff its p.g.fl. satisfies (3.1) for all $h_1, h_2 \in \mathcal{V}_0$.</u>

We shall also be interested in the extended p.g.fl. version of (3.1); this extension is not quite as trivial and is as follows.

PROPOSITION 3.2. <u>When a stationary point process is mixing, its extended p.g.fl. $\overline{G}[.]$ satisfies</u>

$$(3.2) \qquad \overline{G}[h_1 T_x h_2] \to \overline{G}[h_1]\overline{G}[h_2] \quad \text{as} \quad \|x\| \to \infty$$

<u>for all</u> $h_1, h_2 \in \overline{\mathcal{V}}$.

Proof. Start by showing that (3.2) holds for $h_1 \in V_0$, by using a monotone sequence of functions $\{h_{2n}\} \subset V_0$ with $h_{2n}(x) \downarrow h_2(x)$ as $n \to \infty$. Then

$$(3.3) \qquad \bar{G}[h_1 T_x h_{2n}] = G[h_1 T_x h_{2n}]$$

$$\to G[h_1]G[h_{2n}] = \bar{G}[h_1]\bar{G}[h_{2n}]$$

as $\|x\| \to \infty$, and write

$$0 \le |\bar{G}[h_1]\bar{G}[h_2] - \bar{G}[h_1 T_x h_2]|$$

$$\le |\bar{G}[h_1](\bar{G}[h_2] - \bar{G}[h_{2n}])| + |\bar{G}[h_1]\bar{G}[h_{2n}] - \bar{G}[h_1 T_x h_{2n}]|$$

$$+ |\bar{G}[h_1 T_x h_{2n}] - \bar{G}[h_1 T_x h_2]|$$

$$= \delta_{1n} + \delta_{2n}(x) + \delta_{3n}(x) \quad \text{say} .$$

$$\delta_{3n}(x) = |E \exp\int_X \log h_1(y)N(dy)[\exp\int_X \log T_x h_{2n}(y)N(dy)$$

$$- \exp\int_X \log T_x h_2(y)N(dy)]|$$

$$\le |\bar{G}[T_x h_{2n}] - \bar{G}(T_x h_2]| \qquad \text{by monotonicity}$$

$$= |\bar{G}[h_{2n}] - \bar{G}[h_2]| \qquad \text{by stationarity} .$$

Clearly, $\delta_{1n} \le |\bar{G}[h_{2n}] - \bar{G}[h_2]|$, so given $\varepsilon > 0$, we can make both $\delta_{1n} < \varepsilon$ and $\delta_{3n}(x) < \varepsilon$, uniformly in x, by choosing n sufficiently large. Fixing such n, (3.3) implies that for $\|x\|$ sufficiently large, $\delta_{2n}(x) < \varepsilon$ also. Thus (3.2) holds for $h_1 \in V_0$, $h_2 \in \bar{V}$. A similar argument proves it for $h_1 \in \overline{V}$ as well.

4. Mixing of cluster processes .

The cluster processes we consider have the structure

$$(4.1) \qquad N(.) = \sum_{x_i \in N_c} N_m(.|x_i)$$

where N_c is the cluster centre process and the cluster member processes $N_m(.|.)$ are finite point processes coming from an independent family in the sense that for any countable collection of subscripting indices, $N_m(.|x_i)$ are mutually independent a.s. finite point processes which are dependent on N_c only through the locations x_i of the cluster centres. If it is also assumed that

$$(4.2) \qquad N_m(.|x_i) =_d N_m(.-x_i|0)$$

and that $N_c(.)$ is stationary, then it follows that the cluster process $N(.)$ is stationary. While there do exist stationary cluster processes for which $N_c(.)$ is non-stationary and $N_m(.|.)$ does not have the homogeneity property at (4.2), the conditions enunciated around (4.2) do constitute a natural prescription for what we shall always understand by the term <u>stationary cluster process</u>. Writing

$$(4.3) \qquad G_m[h|x] = E \exp\int_X \log h(y) N_m(dy|x) ,$$

it is known (see e.g. Westcott (1971) for references and details) that the p.g.fl. $G[.]$ of $N(.)$ is related to $G_m[.|.]$ and the p.g.fl. $G_c[.]$ of $N_c(.)$ by

(4.4) $$G[h] = G_c[G_m[h|.]]$$

$$= E \exp \int_X (\log G_m[h|x] [N_c(dx)$$

and that

(4.5) $$\int_X (1 - G_m[h|x])N_c(dx) < \infty \quad \text{a.s.,}$$

where $h \in V$. It is to be remarked that for such h and $N_m(.|x)$, it is certainly the case that $G_m[h|x]$ need not be an element of V. To that extent therefore, the right-hand side of (4.4) should be written in terms of the extended p.g.fl. $\bar{G}_c[.]$. In other words, now that an extended p.g.fl. is defined, we can replace the loosely written statement (4.4) by

(4.4)' $$G[h] = \bar{G}_c[G_m[h|.]] \qquad (h \in V)$$

with (4.5) satisfied.

For a stationary cluster process the relation (4.2) has as its p.g.fl. version

(4.6) $$G_m[h|x] = G_m[T_x h|0]$$

which, since the left-hand side equals $T_x G_m[h|0]$, can be written as

(4.6)' $$T_x G_m[h|u] = G_m[h|u+x] = G_m[T_x h|u] .$$

LEMMA 4.1. <u>When the family</u> $\{N_m(.|x):x \in X\}$ <u>satisfies (4.2)</u>, <u>the p.g.fl.</u> $G_m[h|x] \in \bar{V}_0$ <u>when</u> $h \in V_0$, <u>and</u> $G_m[h|x] \to 1$ <u>as</u> $||x|| \to \infty$.

Proof. Let $h \in \mathcal{V}_0$ have $\inf_{x \in \mathcal{X}} h(x) = e^{-\theta} > 0$ for some non-negative finite θ. Then, using (4.6),

$$G_m[h|x] = E \exp\int_{\mathcal{X}} \log h(y+x)\, N_m(dy|0)$$

$$\geq E \exp(-\theta N_m(\mathcal{X}|0)) > 0$$

because $N_m(\mathcal{X}|0) < \infty$ a.s.

For $h \in \mathcal{V}_0$, $T_x h \to 1$ pointwise as $\|x\| \to \infty$ so the convergence to 1 of $G_m[h|x] = G_m[T_x h|0]$ follows from $N_m(\mathcal{X}|0) < \infty$ and part (i) of Theorem 2.1.

The theorem below is in Westcott (1971), though with an incomplete proof. We were led to consider extended p.g.fl.s and the classes $\mathcal{V}_o$, $\overline{\mathcal{V}}_0$, and $\overline{\mathcal{V}}$ through formulating a p.g.fl. proof of the result. An alternative proof is at 11.1.4 of Matthes, Kerstan and Mecke (1978).

THEOREM 4.2. <u>If the cluster centre process of a stationary cluster process as above is mixing, then so is the cluster process.</u>

Proof. In view of Theorem 3.1, Proposition 3.2, and (4.4)', it is enough to show that for all $h_1, h_2 \in \mathcal{V}_0$,

$$(4.7) \qquad G[h_1 T_x h_2] = \overline{G}_c[G_m[h_1 T_x h_2|\cdot]]$$

$$\to \overline{G}_c[G_m[h_1|\cdot]]\overline{G}_c[G_m[h_2|\cdot]]$$

as $\|x\| \to \infty$, it being known that the extended p.g.fl. $\overline{G}[\cdot]$ defined by $G_c[\cdot]$ satisfies (3.2) and that $G_m[h|x] \in \overline{\mathcal{V}}_0 \subset \overline{\mathcal{V}}$ for $h \in \mathcal{V}_0$. Thus, for $h_1, h_2 \in \mathcal{V}_0$

and $\|x\| \to \infty$,

$$(4.8) \qquad \bar{G}_c[G_m[h_1|.]T_x G_m[h_2|.]] \to \bar{G}_c[G_m[h_1|.]]\bar{G}_c[G_m[h_2|.]]$$

$$= G[h_1]G[h_2] \; ,$$

so, writing $\chi_x(u) = G_m[h_1|u]T_xG_m[h_2|u]$,

$$\Delta_x(u) = G_m[h_1T_xh_2|u] - \chi_x(u) \; ,$$

it follows from (4.7) and (4.8) that it is enough to show that as $\|x\| \to \infty$,

$$(4.9) \qquad \bar{G}_c[\chi_x + \Delta_x] - \bar{G}[\chi_x] \to 0 \; .$$

Appealing to the boundedness property noted in Lemma 4.1, $\inf_{x \in X, u \in X} \chi_x(u) = \gamma$ say with $\gamma > 0$, and $\inf_{x \in X, u \in X}(\Delta_x(u)+\chi_x(u)) > 0$, so, as a little manipulation shows,

$$|\bar{G}_c[\chi_x+\Delta_x] - \bar{G}_c[\chi_x]| \leq 1 - E\exp\{-\int_X \log(1+|\Delta_x(u)|\chi_x(u))N_c(du)\}$$

The function $\Delta_x(u)$ is of the form $\mathrm{cov}(X,Y)$ for r.v.s X,Y with the property $0 \leq X \leq 1$, $0 \leq Y \leq 1$, $EX = G_m[h_1|u]$, $EY = G_m[T_xh_2|u] \to 1$ as $\|x\| \to \infty$. For such r.v.s, $|\mathrm{cov}(X,Y)| \leq 1 - \max(EX,EY)$, so $\Delta_x(u) \to 0$ pointwise as $\|x\| \to \infty$, and the relation $|\Delta_x(u)| \leq 1 - G_m[h_1|u]$ shows it to be a.s. N_c-integrable uniformly in x . Now, using the positive bound γ on $\chi_x(u)$ and $|\Delta_x(u)| \leq 1$,

$$|-\log(1+|\Delta_x|/\chi_x)| = |\log(1-|\Delta_x|/(|\Delta_x|+\chi_x))|$$

$$\leq \gamma^{-1}|\Delta_x| \sum_{k=0}^{\infty}(1+\gamma)^{-k} = \gamma^{-2}(1+\gamma)|\Delta_x| \quad .$$

Thus we may apply part (iv) of Theorem 3.1 with $\varepsilon(u) = \gamma^{-1}(1+\gamma)(1-G_m[h_1|u])$ and $|\Delta_x(u)|/\chi_x(u) \to 0$ pointwise as $\|x\| \to \infty$ to conclude that, as $\|x\| \to \infty$,

$$|\overline{G}_c[\chi_x+\Delta_x] - \overline{G}_c[\chi_x]| \leq \overline{G}_c[1] - \overline{G}_c[\chi_x/(\chi_x+|\Delta_x|)] \to 0 \; .$$

(4.9) is established, and the theorem is proved.

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